BOUNDARY LAYER FLOWS PAST DISTINCT STRETCHING SURFACES

SYNOPSIS

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1 Introduction

Fluid mechanics is that branch of science which deals with the study of the static, kinematic and dynamic aspects of fluids (liquids or gases). By fluid statics, we mean the study of fluid at rest, by fluid kinematics we mean the study of fluids in motion when the pressure forces are not considered, and by fluid dynamics we mean the study of fluids in motion when the pressure forces are considered. Fluid mechanics acts as a base for understanding various areas of science and engineering. It is essential to understand the basic principles of fluid mechanics to study any system in which fluid acts as a working medium. The concepts of fluid mechanics are apparent from anything to everything in our daily life. The applications of the different concepts of fluid mechanics can be seen in designing of canal and dam systems, designing of piping systems in chemical plants, aerodynamics of auto-mobiles and flow analysis, designing the cooling system of engines, lubricating systems, transportation etc.

Earth, air, water, fire, and ether are considered to be the five fundamental elements, which this universe is made up of. Out of these five fundamental elements, two (air and water) come in the category of fluids which shows a strong connection with fluids in our life. Hence without fluid flows, life cannot be imagined on earth. Our habitat and the technical world would be very different or it may not exist without fluid flows which show the significance of studying fluid flows. Flow processes occur in many ways in our nature. The fluid is any substance containing particles that can move, change their position, and can easily flow. All kinds of liquids and gases are considered to be fluids. For instance, water, oil, honey, air, etc. are some very basic examples of fluids that matter a lot in our day-to-day life. For example, water plays a vital role for us as it is used for generating electricity, for irrigation in agricultural fields, in thermal plants, as a coolant in auto-mobiles and many more.

1.1 History

Anderson [1] staged the history of fluid dynamics in a refined way. In historic times, many well-known inventors and scientists like Archimedes, Isaac Newton, Leonardo-da-Vinci

gave their remarkable contribution to the study of fluid mechanics. However, in the early 1900s, when Orville and Wilbur Wright invented the first practical plane they needed to calculate the strength and elevation of the plane. The plane had an aerofoil shaped structure like the one in the figure shown below.

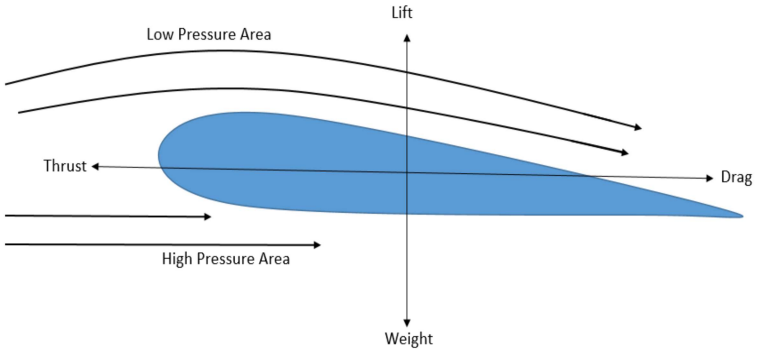

Figure 1: Structure of an aerofoil

The net force on the aerofoil is a result of two forces mainly, fluid pressure and shear stress. Now to calculate the force, one needs to determine both the shear stress and pressure distribution and then one needs to integrate them over the surface of the aerofoil. Earlier in the 20th century, one could determine the pressure distribution using some approximation. The pressure is not much problem to be calculated as in calculating the pressure distribution one can consider the flow to be inviscid but in calculating the shear stress, we need to include the internal friction and viscous flow too. Hence, Orville and Wilbur Wright needed to solve the Navier-Stokes equations which were kind of impossible at that time. Ludwig Prandtl in 1904 gave the concept of boundary layer theory which proved to be a milestone in the history of fluid mechanics. It helped in calculating many factors such as determining the skin friction drag, flow separation, etc. on the body over which the fluid is flowing.

Before Prandtl gave the concept of boundary layer theory, there was serious bewilderment about the role of viscosity in the fluid flow. But Prandtl's work cleared some air. In modern flow analysis, boundary layer theory has come out to be the most historic concept. Prandtl's work showed that the Navier-Stokes equations can be simplified and obtained the boundary layer equations. The boundary layer equations were still in fact coupled and non-linear partial differential equations in nature.

Ancient civilization was very much capable of solving certain flow problems. We know about the irrigation systems from prehistoric times. The equation of continuity in one-dimension for steady flows was first derived by Leonardo Da-Vinci (1452-1519). He was an extraordinary experimentalist. Many of his works contained accurate descriptions of jets, waves, eddy formation, etc. Later, Isaac Newton (1642-1727) gave laws of motion and law of viscosity of the newtonian fluids in 1687. The differential and integral form of equation of motion was developed by Leonhard Euler (1707-1783) which now is known as the Bernoulli Equation. But he decided to ignore the effects of friction acting on the motion. Euler developed two coupled, non-linear partial differential equations which consisted of pressure and velocity flow fields.

D'Alembert (1717-1783), known for developing the mathematics of partial differential equations, then used these equations to develop a famous paradox which states that "The drag force is zero for anybody immersed in an inviscid potential and incompressible flow". Mathematicians of the 18th century gave nice solutions for frictionless flow problems but many engineers rejected their theory and solutions because frictionless flows had very limited applications in real life, then the science of hydraulics came into existence which was mostly relied on experiments. In 1883, Lord Rayleigh (1842-1919) proposed the technique of dimensional analysis and Osborne Reynolds (1842-1912) showed the significance of the dimensionless number which was named after him as Reynolds's number.

After almost a century, Euler's equations were modified which incorporated the effects of internal friction in a flow field which resulted as a system of non-linear partial differential equations, which are today known as Navier-Stokes equations. These equations were derived in 1822 by Claude-Louis Navier. The derivation by Stokes in 1845 was the first to explicitly introduce viscosity. However, it was almost impossible to obtain the analytical solutions of these equations. Orville and Wilbur Wright invented the first practical airplane in 1903 and needed to calculate the drag and lift on the airplanes. To calculate the force, they first needed to determine the pressure and the shear stress distributions and then integrate them over the surface. Pressure distribution was somewhat easier to calculate as in calculating pressure distribution we can assume the fluid to be inviscid and frictionless but in calculating the shear stress distributions one must consider viscous flow include the internal friction. This, in turn, meant to work out the

Navier-Stokes equations which were kind of undoable at that time.

1.2 Prandtl's boundary layer

However, in 1904, Ludwig Prandtl [2], a german scientist introduced the notion of boundary layer. He speculated that the effect of viscosity is dominant only in the very narrow vicinity of the solid surface boundary over which the fluid is flowing, which is generally known as a no-slip condition.

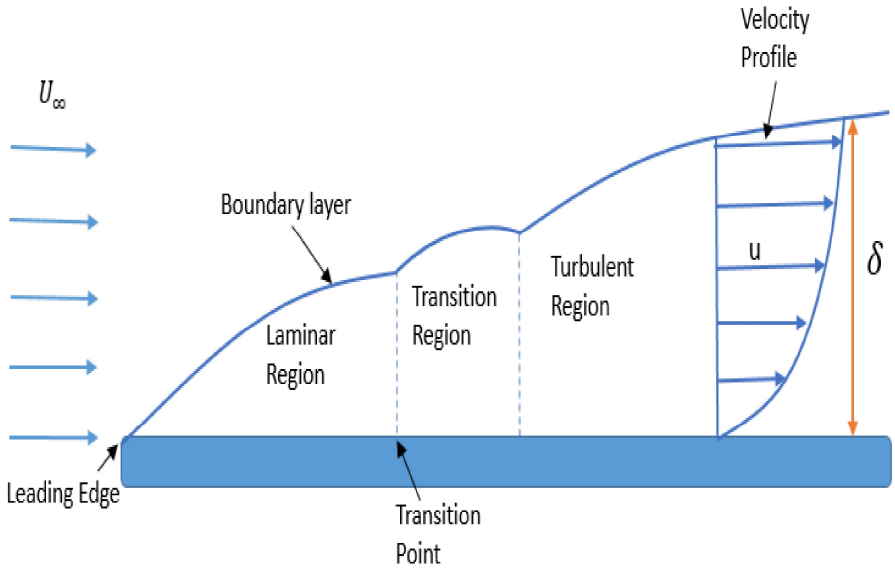

Figure 2: Schematic diagram of a boundary layer

Let us consider a fluid flow past a solid flat plate. Using theoretical considerations Prandtl divided the flow of the fluid into mainly two parts:

1. **Boundary layer flow** $(0 \leq y \leq \delta)$: A thin layer fringing the vicinity of the solid surface boundary where the viscous effect is dominant and relative velocity exists.

2. **Outer region flow** $(y > \delta)$: The outer region where no relative velocity exists and the viscous forces are negligible.

The boundary layer can also be expressed as a layer within which a relative motion among the fluid particles exists. The outer flow can also be expressed as a potential flow.

Prandtl's work reduced the Navier-Stokes equations to a simpler form, now known as boundary layer equations. These simpler boundary layer equations were solved by one of Prandtl's students, Hartman Blasius [3] for laminar flow over a flat plate which resulted in equations for skin friction drag and boundary layer thickness.

Falkner and Skan [4] extended the two-dimensional Blasius flow for a wedge flow. Many renowned researchers further extended the problem for three-dimensional and axisymmetric flows also. All these flows were studied assuming that there exists external potential flow whatever be the case, i.e. whether the surface is static or moving. The opposite case i.e. the case where the fluid is static, for two-dimensional and axisymmetric flow was first investigated by Sakiadis [5]. This case was just the opposite of the Blasius flow. In his investigation, Sakiadis found that some physical parameters had different values in this case. For example, viscous drag was considerably greater in this case in comparison to the Blasius flow which showed that Blasius solutions cannot be applied in this specific case and hence they are to be investigated independently.

However, boundary layer equations showed completely different characteristics than Navier-Stokes equations. Navier-Stokes equations showed elliptic behavior whereas boundary layer equations showed parabolic behavior. Upto a certain value of Reynold's number, we get smooth solutions of Navier-Stokes equations whereas beyond this certain value of Reynold's number, we get an unstable turbulent flow. In fluid dynamics, Reynolds's number is represented as the proportion of the inertial forces to the viscous forces within a fluid and thus Reynold's number is a dimensionless quantity. It would not be unsuitable to say that the Reynold's number depict the relative strength of inertial forces to viscous forces. Reynold's number comes handy in predicting the various features of the fluid flow. High value of Reynold's number represents small viscosity and turbulent flows with high swirling motions called eddies whereas low value of Reynold's number represents high viscosity and laminar flow. One of the most historic applications of Prandtl's work was the calculation of many factors like skin friction drag, pressure drag, flow separation, etc. on the body over which the fluid is flowing. These were some of the earliest achievements of Prandtl's work.

1.3 Boundary layer equations

For a two-dimensional flow, the full equation of motion for a viscous incompressible fluid in the absence of body forces is given as (Schlichting [6])

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

where u and v are the velocity components, x and y are horizontal and vertical coordinates respectively, p is the pressure, t is the time, ρ is the density and ν is the kinematic viscosity. xy-plane is the plane of motion, x coordinate is along the solid boundary and y coordinate is perpendicular to it. Here we will obtain the asymptotic form of the Navier–Stokes equations at very high Reynold's number. Let x', y', t', p', u' and v' be dimensionless quantities such that

$$x'=\frac{x}{X},y'=\frac{y}{Y},t'=\frac{t}{T},p'=\frac{p}{P},u'=\frac{u}{U},v'=\frac{v}{V}, \tag{4}$$

where X,Y,T,P,U and V are the units of measurement of relative quantities.

From equations (1),(2),(3) and (4), we get

$$\frac{X}{UT}\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial u'}{\partial y'}=-\frac{P}{\rho U^2}\frac{\partial p'}{\partial x'}+\frac{\nu}{XU}\frac{\partial^2 u'}{\partial x'^2}+\frac{\nu X}{Y^2U}\frac{\partial^2 u'}{\partial y'^2}, \tag{5}$$

$$\frac{X}{UT}\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial v'}{\partial y'}=-\frac{PX}{\rho YUV}\frac{\partial p'}{\partial y'}+\frac{\nu}{XU}\frac{\partial^2 v'}{\partial x'^2}+\frac{\nu X}{Y^2U}\frac{\partial^2 v'}{\partial y'^2}, \tag{6}$$

$$\frac{\partial u'}{\partial x'}+\frac{XV}{YU}\frac{\partial v'}{\partial y'}=0. \tag{7}$$

We shall obtain an estimate for the above equations in terms of Reynold's number. Let us treat X and U as the basic units so that Reynold's number for the flow is demonstrated

as

$$Re = \frac{XU}{\nu}. \tag{8}$$

We can also express units of pressure and time in terms of X and U as

$$T = \frac{X}{U} \qquad \textit{and} \qquad P = \rho\, U^2. \tag{9}$$

Viscous forces and inertia are of the same order in the region of boundary layer as they balance each other inside the boundary layer. If we consider a single flow parameter *Re*, we must have

$$\frac{\nu}{UX}\left(\frac{X}{Y}\right)^2 = O(1) \qquad \textit{and} \qquad \frac{XV}{YU} = O(1)$$

$$\implies \qquad Y = O\left(\frac{X}{\sqrt{Re}}\right) \tag{10}$$

The dynamic similarity in the two flows at points where viscosity is not important is certified by number $P/\rho U^2$ and in the vicinity of the boundary where viscosity is dominant, the dynamic similarity is certified by the number ν/XU. By dynamical similarity we mean that the two different incompressible viscid flows flowing past similar geometrical surface have same value of Reynold's number.

Now with equation (10), if we ignore the primes in the non-dimensional governing equations, we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{11}$$

$$\frac{1}{Re}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{1}{Re^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{13}$$

When $Re \longrightarrow \infty$, the above equations reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{14}$$

$$0 = -\frac{\partial p}{\partial y}, \tag{15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{16}$$

From equation (15), we see that p is a function of x only, which shows that pressure is constant in the direction normal to the boundary surface. It might be same as the pressure outside the boundary layer where we judge it by the inviscid flow.

Therefore,

$$-\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x},$$

where U is the free stream velocity.

Since the boundary surface is non-porous and there applies no-slip condition, the velocity components u and v both must vanish at $y = 0$.

Hence to solve the Prandtl's boundary layer equations, the boundary conditions are

$$u = v = 0 \qquad \text{w}hen \qquad y = 0$$

$$u = U \qquad \text{w}hen \qquad y \longrightarrow \infty$$

For a two-dimensional unsteady incompressible flow, the Prandtl's boundary layer equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

1.4 Boundary layer separation

The one factor which affects the fluid flow substantially is the pressure gradient. A positive pressure gradient is denominated as an adverse pressure gradient whereas a negative pressure gradient is denominated as a favorable pressure gradient. An adverse pressure

gradient retards the fluid flow whereas a favorable pressure gradient enables the fluid flow. An adverse pressure gradient is hard to ignore however we can overcome the effects by offering a more favorable pressure gradient. Austere aftermath that it implies on fluid flow is to separate the flow. Separation usually occurs when a flow turnabout occurs. In aerodynamics, separation is typically unwanted because it boosts drag. It is not possible to change the fluid's viscosity but it is possible to hold back the separation in many other ways.

When a fluid flows past a body or a body is fully enwrapped in the fluid, a thin boundary layer is formed on the surface of the body. Inside this boundary layer, relative flow exists and potential flow exists outside the boundary layer region. The velocity of the fluid particles varies in the direction perpendicular to the body from zero to the free stream velocity. Since boundary layer thickness is a function of x, so the thickness of the boundary layer also increases along the length of the boundary layer. Along the length of the boundary layer, more fluid particles from the external flow enter the boundary layer region, and the decelerated fluid particles inside the boundary layer are forced to move out, and consequently, the boundary layer separates from the surface. The threshold point on which the boundary layer is at the cusp of separation is denominated as a point of separation and the whole phenomena is called boundary layer separation.

So basically the pressure difference brings the drag force, particularly known as form drag and the shear stress brings the skin friction drag. The outcome of these two forces collectively results as a drag on a body.

Flow separation happens when:

- An adverse pressure gradient happens in the direction of fluid flow.
- The velocity at the plate is negative or zero
- An inflection point subsists in the velocity profile.

One can understand the concept of separation of boundary layer flow as:

The pressure gradient plays a vital role in the formation of boundary layer separation. Now suppose that the fluid is flowing against the pressure gradient. Pressure over the breadth of the boundary layer is the same as outside the boundary layer. The flow gets

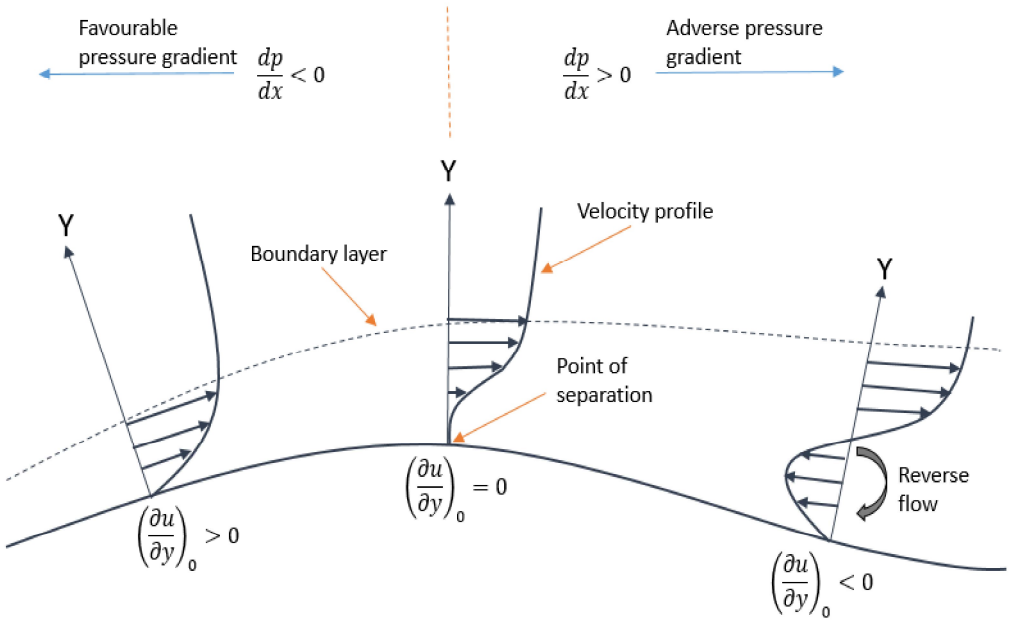

Figure 3: Schematic diagram of a flow separation

accelerated when we offer a favorable pressure gradient or negative pressure gradient i.e. when the pressure decreases in the direction of the flow. The force due to the action of viscosity and pressure forces add together and tend to reduce the effect of friction in the boundary layer which results in the decrease of the boundary layer thickness in the direction of the flow.

However, the flow is decelerated when we offer an adverse pressure gradient or positive pressure gradient i.e. when the pressure increases in the direction of the flow. Here the friction at the boundary wall and an adverse pressure gradient both increase the hindered effect of viscosity. Hence, the boundary layer thickness increases in the direction of the flow. The momentum in the boundary layer region is already lower than the regions of free stream velocity. The continuous deceleration of the flow brings it to rest at some point and after this point, the flow reverses. Here a region of flow recirculation develops and we say that the flow is separated.

1.5 Boundary layer thickness

The velocity of the fluid particles within the boundary layer region increases asymptotically from zero to 99 percent of free stream velocity U_∞. The distance over

which the velocity of the particles reaches the edge of the boundary where the velocity is 99 percent of free stream velocity is defined as the boundary layer thickness.

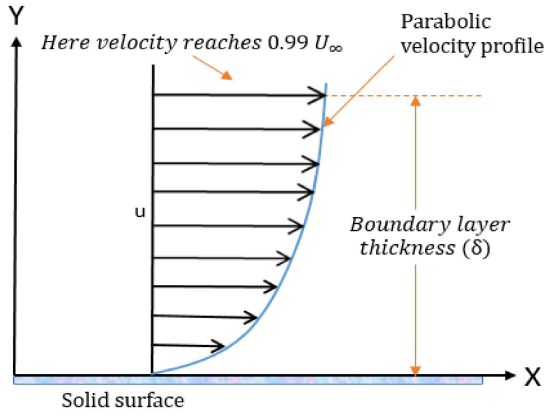

Figure 4: Boundary layer thickness
Here U_∞ denotes the free stream velocity.

Since the definition of the boundary layer thickness is arbitrary to some extent, the other three kinds of thicknesses based on physical measurements are defined as:

1.5.1 Displacement thickness

Displacement thickness is defined as the distance measured perpendicular to the solid boundary by which the solid boundary must be displaced to compensate for the reduction in mass flow rate.

Mathematically, displacement thickness is given as

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

1.5.2 Momentum thickness

Momentum thickness is defined as the distance measured perpendicular to the solid boundary by which the solid boundary must be displaced in order to compensate for the reduction in momentum or change in momentum.

Mathematically, momentum thickness is given as

$$\delta_2 = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

1.5.3 Energy thickness

Energy thickness is defined as the distance measured perpendicular to the solid boundary by which the solid boundary should be displaced in order to compensate for the reduction in kinetic energy.

Mathematically, energy thickness is given as

$$\delta_3 = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u^2}{U_\infty^2}\right) dy$$

1.6 Drag and lift

If a solid body is immersed entirely in the flow region of fluid flow then it is supposed to endure two kinds of forces, generally known as drag and lift. Drag is the force component applied on the body in the direction of the free-stream of the fluid. Lift is the force component applied on the body perpendicular to the free-stream of the fluid. Both are affected by the shape of the object and Reynold's number of the fluid flow. In the mathematical form we define them as:

$$D = Drag = \frac{C_D A \rho U_\infty^2}{2}$$

and

$$L = Lift = \frac{C_L A \rho U_\infty^2}{2}$$

The drag force is usually due to shear forces and pressure acting on the surface of the body.

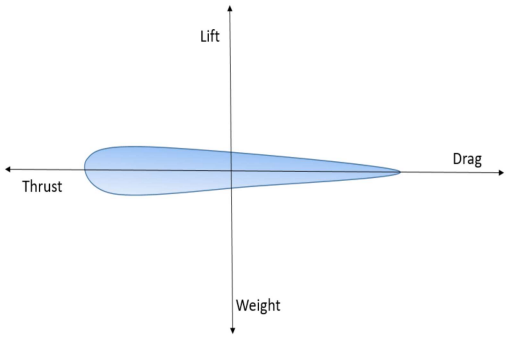

Figure 5: Lift and drag on a body

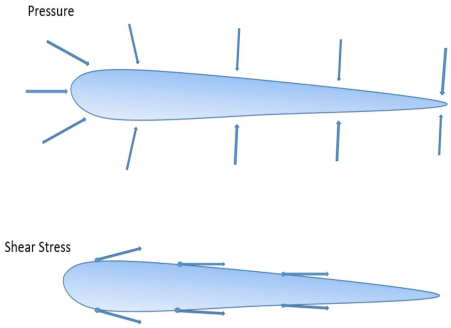

Figure 6: Pressure and shear stress distribution on a body

1.6.1 Local skin coefficient

If τ_w is the shearing stress on the boundary surface given as $\tau_w = \mu(du/dy)_{y=0}$, then the local skin coefficient is defined as the dimensionless shear stress given as

$$C_f = \frac{\tau_w}{(\rho U_\infty^2/2)}$$

1.7 Steady and unsteady boundary layers

A flow in which the flow properties such as density, viscosity, temperature, velocity, pressure, etc., vary with respect to time is demonstrated as steady flow whereas a flow in which these flow properties do not vary with respect to time is demonstrated as unsteady flow. Some of the unsteady flows become steady under certain circumstances and are known as pseudo steady flows. Most of the start-up processes are unsteady in nature, for instance, water running out of a tap is initially unsteady but soon it becomes steady.

One of the examples of steady motion is the flow of water in a pipe of variable diameter at constant pressure. Some examples of unsteady motion are lane of a flood wave, high tidal effects, the flow of water in a pipe of variable diameter, and variable pressure because of an elevation and decline in water levels, flow through turbo-vehicle, etc. There is a wide literature available on steady boundary layer problems because of its great practical applications like lubricating oils in engineering etc. Blasius flow over a flat plate is the most fundamental example of a steady boundary layer flow. In his study, Blasius considered a two-dimensional steady and incompressible flow over a flat plate with an angle of incidence equal to zero. Both steady and unsteady boundary layer flows has a vital role to play in various applications in engineering, science, and technology. However, in the beginning, the boundary layer theory was assumed for only steady flows but before long it was discovered that many complex boundary layer problems were in fact allied unsteady flow.

The Prandtl's boundary layer equations for a 2-dimensional unsteady incompressible flow are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{\partial^2 u}{\partial y^2},$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

1.8 Slip and no-slip boundary conditions

Navier-Stokes equations are highly non-linear partial differential equations which on integration give some constants that have to be evaluated to obtain flow field's exact solution. These solutions are obtained after changing some suitable boundary conditions which are analyzed based on physical observations. In boundary layer theory, no-slip condition plays the most important role. No-slip condition refers to the circumstances when the velocity of the fluid adjacent to the boundary has the same velocity as that of the boundary. If the boundary is static, the velocity of the fluid adjacent to the wall will be zero. If the boundary is in motion, the fluid adjacent to the wall will have velocity the same as that of the boundary.

When the no-slip boundary conditions do not apply, slip conditions have to be assumed.

Partial slip condition generally refers to the non-zero velocity conditions. When the fluid at the boundary does not exhibit the same velocity as that of the boundary, the fluid is said to slip along the boundary. This kind of flow is also known as slip flow.

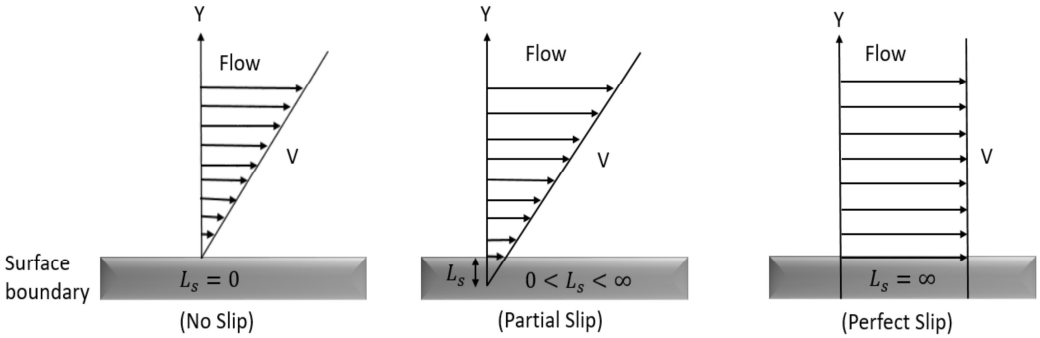

Figure 7: Slip and no-slip boundary conditions

There is another term known as a perfect slip. A perfect slip condition is achieved when the boundary surface does not affect the velocity profile of the fluid flowing. The discrepancy between the velocity of the boundary and the fluid adjacent to the boundary is demonstrated as a slip velocity denoted as V_s.

Mathematically,

$$V_s = \frac{\mu}{C_s} . \frac{\delta V}{\delta y}$$

where μ is the coefficient of viscosity, C_s is the coefficient of slip, and $\frac{\delta V}{\delta y}$ is the velocity gradient.

Slip length or the measure of slip, denoted as L_s, is defined as an extrapolated distance below the surface where the tangential velocity component vanishes. Factors like shear stress, roughness of the surface, viscosity affect the measure of slip directly.

$$L_s = \mu / C_s$$

The perfect slip condition is given as $C_s = 0$ and the no-slip condition is given as $C_s = \infty$.

1.9 Flow past a stretching sheet

Fluid dynamics is based on three basic principles of mass, momentum, and energy and the application of these basic principles to different kinds of flows over various kinds of

surfaces is undoubtedly very challenging. In our surroundings, fluid flows over differently shaped bodies occur very frequently and promote many physical phenomena, for instance, dust particles in the wind, pipelines underwater, transportation of blood cells in the body, wind flow over a building, vehicles moving through the air, etc. Such kinds of flows are known as external flows. The shape of the body over which the fluid flows, play a very vital role in the study of the characteristics of heat and mass transfer.

An important kind of flow is one on moving solid surfaces as they have applications in many technical processes. The study of boundary layer flow due to different stretching surfaces has a broad range of applications in industrial and technological processes. An example of a moving continuous surface is a polymer sheet ejected continuously from a die. This notion is applied in industries to elongate plastic films.

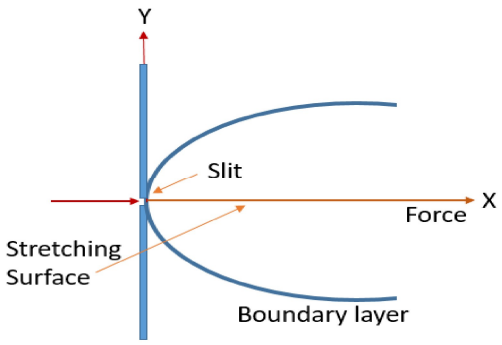

Figure 8: Schematic diagram of a stretching sheet

The flow generated because of the stretching of the resilient flat surface after the employment of stress, moving in its plane, and velocity deviating with the distance from a particular point is known as Stretching flow. This kind of flow has never been studied thoroughly especially the cases of axisymmetric and three-dimensional flows past stretching surfaces. Also, there has been less investigation of non-similar and turbulent flows.

Sakiadis was the first to investigate this problem. Sakiadis studied a long continuous sheet coming out of a slit whose boundary layer thickness elevates as the distance step-ups. In his work, Sakiadis showed the difference between the behavior of the boundary layer on a moving surface and a finite length surface. Crane [7] first extended the Sakiadis problem of flow for which stretching of sheets was responsible when it follows the

power-law profile or the velocity deviates linearly. Many researchers extended Crane's problem for different physical conditions. The problems investigated by Crane and other researchers comprehended the cases where the velocity of the stretching sheet is directly proportionate with the distance from a fixed point. However, this relationship may not be linear all the time. A surface may stretch non-linearly, exponentially, etc. Sakiadis flow problem was extended by Magyari and Keller [8] for exponential wall velocity. These flows are known as stretching sheet flows. Similar solutions for these cases have been explored for two-dimensional flows and very few have explored the three-dimensional and axisymmetric flow cases. Study on non-similar flows because of stretching surfaces is also aberrant for cases like two-dimensional, three-dimensional, and axisymmetric flows.

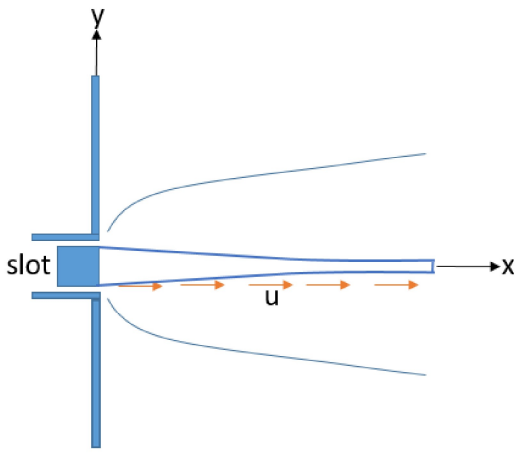

Figure 9: Schematic diagram of a flow past surface of variable thickness

1.10 Nanofluids

A constructed colloidal solution made up of a base solution and nano-particles is known as a nanofluid. Nanoparticles are basically nano sized particles (i.e. having size between 1-100 nano-meters) of metallic and non-metallic origin. For the configuration of nanofluids, the most familiar nanoparticles used are aluminium (Al), silicon (Si), titanium (Ti), magnesium (Mg), iron (Fe), zinc (Zn), graphene, diamond, carbon nano-tubes etc. and the most familiar base solutions used are conductive liquids like ethylene glycole, water, ethanol, bio-fluids, oils etc. Nanofluids have gained a lot of attention in the recent times because of its properties like higher thermal conductivity, energy transportation,

thermal diffusivity, viscosity, specific heat etc. In the configuration of nanofluids, addition of mere 1-5 vol% of nano particles in the base fluid can improve the effective thermal conductivity of the base fluid by 20% however this improvement in the effective thermal conductivity also gets affected by various factors like dimensions of the particle, shape of the particle, concentration of nanoparticles added, thermo-physical properties of the particles, type of the base fluid, temperature etc. [9].

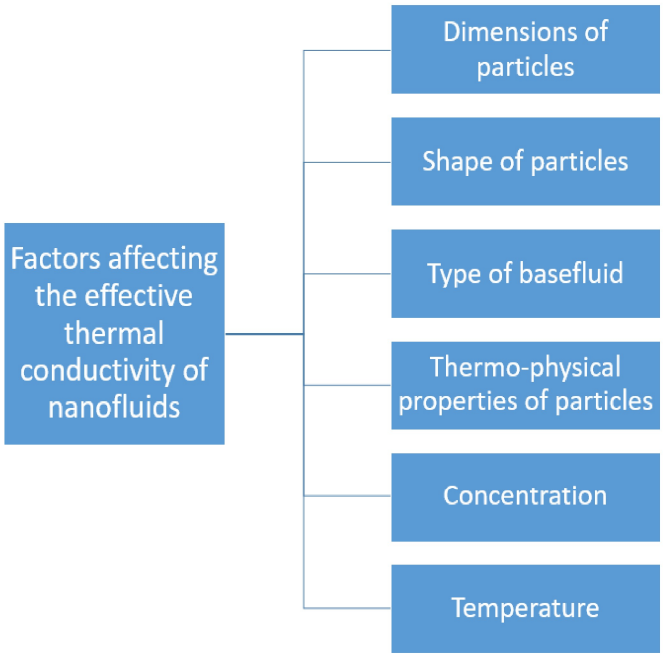

Figure 10: Factors affecting the effective thermal conductivity of nanofluids

Many researchers ([10]-[15]) investigated the heat transfer behaviour of nanofluids in their research because of its endless industrial applications in transportation, auto-mobiles, energy production, bio-medical areas, electronics etc. The never ending applications of nanofluids have made it the most discussed research area in last few years and its studies are being utilized to identify and develop more superior coolants.

2 Literature review

Potential flow theory was able to calculate the airflow and pressure distribution around different shapes of bodies. It assumed the flow to be incompressible, inviscid, and

irrotational. But it was the failure of the potential flow theory to calculate the drag on the bodies over which the fluid flows because of the assumption of zero relative velocity between the wall of the body and the fluid layer. Ludwig Prandtl [2] coined boundary layer theory by presenting a paper on the motion of a fluid with very little viscosity at the Third International Congress of Mathematicians in 1904 at Heidelberg. Prandtl stated something which was never suggested before by anyone and thus unclogged the perception of real fluids [16]. Prandtl was the one who first introduced the term no-slip condition. He stated that the viscous effects were to be felt only in the very close vicinity of the surface over which the fluid is flowing. Also, a very agreeable description of the boundary layer was given by Prandtl as he framed the boundary layer to be a result of adhesion of the fluid to the walls of the body and if the viscosity of the fluid is less and the fluid path is not too long, then the velocity of the fluid will converge to its free stream velocity in a very short distance [1].

The study of fluid flow over a stretching sheet has gained tremendous popularity in recent times because of its applications in industrial areas [6]. Boundary layer flow over a stretching sheet has its applications in many areas of engineering like fibre industries, the coating of emulsion on photographic films, manufacturing of plastic films, etc. [17]. In industries dealing with plastics and metals, the flow over a stretching body is important in processes like manufacturing processes, extrusion processes, condensation processes, etc. [18]. During the manufacturing process, a stretched sheet interacts with the fluid both thermally and mechanically [19]. The resulting product is substantially dependent upon the heat transfer mechanism and the flow [20]. So in order to gain some basic understanding of the process of continuously stretching sheets or surfaces, we need to analyze the momentum and heat transfer in such processes. Improvement in the heat transfer during the processes like polymer extrusion, industries of textiles, industries of papers, manufacturing of wires, manufacturing of artificial films, etc. can deprecate the timing of manufacturing and simultaneously it can upgrade the life of the machine ([21], [22]). All the studies on the heat convection and boundary layer flow past an exponentially stretching surface has been done assuming the no-slip condition.

Crane [7] was the first who attained the closed-form solution of a steady flow of a fluid which is incompressible and viscous in nature over an elastic sheet where the flow is being caused by velocity varying linearly with the length from a fixed juncture. Carragher

and Crane [23] and Gupta and Gupta [24] were the next who inquired about the transfer of heat in the flow over-stretching surfaces at a fixed temperature as well as at variable temperature. A special case of suction and blowing was discussed by Chen and Char [25]. Many researchers like Dutta et al. [26], Vajravelu [27], Mukhopadhyay et al. [28], Ishak et al. [29], extended the problem discussed by Crane for different outlooks of flow and heat transfer properties when the surface is stretched linearly.

In most of the studies, it is assumed that the properties of the fluid are constant but it is a well-known fact that most of the thermophysical properties of the fluids are temperature-dependent, that is, the thermophysical properties of fluids may change with temperature, especially the thermal conductivity of the fluids. Under certain circumstances, where the temperature gradient exists across the fluid medium, the thermophysical properties of the fluid may change significantly. Also, it is known that the viscosity of the liquids decreases with an increase in temperature whereas the viscosity of the gases increases with an increase in temperature. Batchelor's viscosity model [30] was studied by Mukhopadhyay et al. ([31]-[32]) which included the flow of a viscid fluid coming as a result of a stretching surface in the presence of a magnetic field.

However, many researchers like Ali [33], Vajravelu [34], Vajravelu et al. [35], Batallar [36] and Cortell [37] investigated boundary layer flow over non-linearly stretching surfaces as all stretching surfaces may not be linear. Vajravelu [34] encountered that a stretching surface may be non-linear. Heat transfer past a stretching sheet has many applications in industries as a result of which many researchers have studied laminar flow and heat transfer over stretching surfaces. Flow over an exponentially stretching surface also has a broad diversity of applications in industries and technology. It was Magyari and Keller [38] who first foresighted the transfer of heat and mass on boundary layer flow due to an exponentially stretching sheet or surface. El-Aziz [39] explained the conduction of heat and flow due to an exponentially stretching sheet. Further many researchers extended the problem for different cases like Elbashbeshy [40] considered the case of an exponentially stretching porous surface. Khan [41] and Sanjayanad and Khan [42] explored the case of viscous-elastic heat transfer and boundary layer flow as a result of exponentially stretching surface. Subsequently, Sajid and Hayat [43], using analytical solutions, studied the impact of thermal radiation over the boundary layer flow as a result of the exponentially stretching sheet. Afterwards, using numerical solutions, Bidin and Nazar [44] studied the impact of

thermal radiation on heat transfer and steady boundary layer flow over an exponentially stretching sheet. Pal [45], in the presence of a magnetic field, studied the boundary layer flow past an exponentially stretching sheet. However, the effects of the magnetic field on the heat convection and boundary layer flow past an exponentially stretching surface was investigated by Ishak [46].

Many researchers like Bhattacharyya [47] have investigated some different flow problems under particular circumstances past a stretching sheet assuming slip/velocity slip conditions. The fluids demonstrating slip conditions find many applications in different medical areas. In all the above-discussed problems where distinct cases of the flow due to stretching sheets are considered, the flow field is assumed to be at steady state. Even so, this is not the case every time. Under certain circumstances, the flow field can be unsteady in nature. For when the surface is stretched abruptly, the flow field can be unsteady. Researchers like Andersson et al. [48], Dandapat et al. [49], Ali and Magyari [50], and Dandapat et al. [51] used the similarity approach to investigate the case for the unsteady isothermal stretching surface. The similarity solutions for the problem of the heat transfer and unsteady flow past an unsteady stretching sheet were discussed by Elbashbeshy and Bazid [52].

For a viscous and incompressible fluid, the problem was investigated by Sharidan et al. [53]. Further, the thermal characteristics of a liquid film caused by an unsteady stretching surface with varying temperatures were studied by Liu and Andersson [54]. However, the similarity solutions for the boundary layer equations of heat transfer and unsteady flow past an unsteady stretching sheet or surface under distinct certain conditions were investigated by many researchers like Tsai et al. [55], Chamkha et al. [56], and Mukhopadhyay ([57]-[59]).

In most of the problems discussed above, the stretching sheets are flat having no curvature. But there are many applications of flow due to a curved stretching surface in an industrial area. But the problems involving flow due to a curved stretching surface are very complex because of the sharp curvature interacting. The mathematical modeling for these kinds of problems does not contain cartesian coordinates. Sajid et al. [60] were the first who managed to obtain the mathematical modeling for boundary layer flow due to a curved stretching sheet or surface owing to uniform curvature. The governing equations were

obtained using a curvilinear coordinate system. Sajid et al. [61] extended this problem for micropolar fluid. Abbas et al. [62] investigated the analysis of heat transfer and laminar flow past a curved stretching surface as for electrically conducting viscous fluid.

A quite fascinating flow demeanor was encountered for negative power indices when the analytical solutions were found for different velocity power indices in the study of boundary layer flow past a stretching surface of variable thickness by Fang et al. [63]. A solution for a boundary layer flow of a newtonian fluid resulted from a non-linear stretch of a surface of variable thickness including slip velocity and power-law surface velocity was discussed numerically by Khader and Megahed [64] using the Finite Difference Method (FDM). Hayat et al. [65] addressed the case of boundary layer flow caused by stretched non-linear surface of variable thickness and repercussions of homogeneous and heterogeneous reactions in it. A boundary layer flow from stretched sheet of variable thickness of fractional Maxwell fluid was investigated by Liu and Liu [66].

Choi [67] who introduced the term nanofluids in 1995, found that uniform and stable suspension of a certain quantity of nanosized particles in a base fluid can enhance the thermal properties of the base fluid amazingly and thus demonstrated that nanofluids show superlative thermal properties in comparison to the conventional fluids. Although the idea was first discussed by Maxwell [68] but the first observations to improve thermal conductivity in liquid dispersions of submicronic particles were developed by Masuda et al. [69] in 1993 where he encountered the hardship of settling down of the particles in the fluid. However, the study of nanofluids for practical applications have recently become widely available to us. The first document on stretching film in nanofluids was given by Khan et al.[70] in which he examined the problem of laminar fluid flow due to stretching of flat surface in a nanofluid, numerically. The flow of the boundary layer, which is induced in a nanofluid due to a linearly stretching sheet was examined numerically by Makinde et al. [71]. Mustafa et al. [72] reported the flow of a nanofluid near a stagnation point toward a stretching surface by incorporating the effects of brownian motion and thermophoresis and developed the solutions using the Homotopy Analysis Method (HAM). Similarly, considering the effects of both brownian motion and thermophoresis, an analytical solution was given by Hassani et al. [73] for boundary layer flow of nanofluids over a stretching sheet. An analysis was carried out by Vajravelu et al. [74] to investigate the convective heat transfer in a stream of nanofluids on a stretching surface. The work was focused

on Ag-water and Cu-water nanofluids, investigating the effects of volume fraction of nanoparticles under the influence of buoyancy and temperature-responsive internal heat absorption and generation on the flow and heat transfer properties. Das [75] inquired for the same problem past a permeable stretching surface considering partial slip. Same case for unsteady boundary layer flow past stretching/shrinking surface was studied by Bachok et al. [76]. Mustafa et al. [77] discussed the two-dimensional steady nanofluid flow resulting from non-linearly stretching sheet by employing convective surface boundary conditions in their work. Mabood and Khan [78] carried out an analytical solution for a two-dimensional MHD unsteady nanofluid flow over a hot surface.

3 Research gap identification

Within the framework of boundary layer theory, boundary layer flow past distinct stretching surfaces is a critical topic which needs more exposure in future. The prime motivation for picking this area as my research area is its never ending applications in many industrial, manufacturing, auto-mobiles, transportation, extrusion processes etc. The base of this very research will be obtaining the demeanor of the boundary layer flows of nanofluids past distinct stretching surfaces.

4 Objectives

The leading objectives of this research work will comprise of the following:

- To employ Buongiorno/ Tiwari and Das nanofluid model for obtaining the mathematical models of the considered physical problems.
- To find the self-similar solutions of the established problems by employing the similarity transformation.
- To find similar solutions using fast and precise numerical methods. To generate a computer code in MATLAB/ MATHEMATICA/ MAPLE for the incorporated numerical method.

- To analyze the impact of physical factors like permeability of porous medium, strength of magnetic field, chemical reaction, radiations, thickness parameters on the nanofluid flows past exponential stretching, linear, nonlinear stretching surfaces and stretching surface with variable thickness.

- To agitate in detail, the obtained results in the form of graphs and tables.

- To comprehend the role of geometries (linear/ non-linear/ exponential/surface of variable thickness) subjected to stagnation point flow of nanofluids.

5 Methodology

Mathematics is a language of science and engineering. It is capable of providing fine expression for any concept. Non-linear differential equations are used to model many laws in science and engineering. The past few decades have seen a remarkable evolution in the area of non-linear equations. Boundary layer equations have proved to be one of the arch reasons in the development of study on non-linear equations. The notion of the boundary layer was first given by Prandtl in which he described the boundary layer to be a very thin layer adjacent to the wall where the viscosity effects are significant however high the Reynolds number of the fluid flow may be. The boundary layer equations were derived using Navier-stokes equations and boundary layer approximations. However, these equations can be converted into non-linear coupled ODEs/PDEs using similarity transformations. But solving these equations is quite a task. Such equations demand a numerical scheme which is fast and precise. Keller-box method is one such scheme which is capable of giving numerical solutions to non-linear cases over some fixed region in the province of the problem. Keller-box method is an implicit finite difference scheme advanced by Cebeci and Bradshaw in 1984 which is very popular nowadays because of its accuracy and convergence.

We will follow the following steps to obtain the solution:

1. First, we will derive the governing boundary layer equations.

2. We will then execute similarity transformation to find the similar solutions.

3. This non-dimensional structure will be solved numerically using Keller-box method with the help of a software MATLAB using the following steps:

   (a) We reduce the system of non-linear differential equations into a system of first order differential equations.

   (b) We further substitute the functions by mean values and their derivatives with central difference formulas and as a result we will get a system of non-linear algebraic difference equations where the number of unknowns is equal to the number of difference equations and boundary conditions.

   (c) Then we use Newton's linearization technique to linearize the non-linear terms in the difference equations.

   (d) Lastly, Block tri-diagonal scheme is used to solve the resultant system.

4. The results will be graphically interpreted for different values of dimensionless parameters.

6 Thesis outline

Considering the above objectives in mind, chapters 2, 3, 4 and 5 will be constructed.

Tentative chapter plan of this thesis is as under:

Chapter 1	Introduction
Chapter 2	Literature Review
Chapter 3	Boundary layer flow due to a linearly stretching surface
Chapter 4	Boundary layer flow past an exponentially stretching surface
Chapter 5	Boundary layer flow over a stretched surface of variable thickness
Chapter 6	Conclusions and future work

Research plan schedule:

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